Appendix D: Mathematical Modeling

The CA CCSSM include mathematical modeling as a Standard for Mathematical Practice (MP.4: “Model with mathematics.”), which should be learned by students at every grade level. In higher mathematics, Modeling is a conceptual category; the modeling standards are spread throughout the other conceptual categories, a star symbol (⋆) indicating that they are modeling standards. This appendix serves to clarify the meaning of mathematical modeling and the role of modeling in teaching the CA CCSSM.

What isn't mathematical modeling?

The terms "model" and "modeling" have several connotations, and while the term "model" has a general definition of "using one thing to represent something else," *mathematical modeling* is something more specific. Below is a list of some things that are *not* mathematical modeling in the sense of the CA CCSSM.

- It is not modeling in the sense of, "I do; now you do."
- It is not modeling in the sense of using manipulatives to represent mathematical concepts (these might be called "using concrete representations" instead.)
- It is not modeling in the sense of a "model" being just a graph, equation, or function. Modeling is a process.
- It is not just starting with a real-world situation and solving a math problem; it is returning to the real-world situation and using the mathematics to inform our understanding of the world. (I.e. contextualizing and de-contextualizing, see MP.2.)
- It is not beginning with the mathematics and then moving to the real world; it is starting with the real world (concrete) and representing it with mathematics.

What is mathematical modeling?

Put simply, *mathematical modeling* is the process of using mathematical tools and methods to ask and answer questions about real-world situations (Abrams 2012). Modeling will look different at different grade levels; success with modeling will be based on students' mathematical background knowledge and ability to ask modeling questions. But all mathematical modeling situations share similar features, which will be discussed below. For example, at a very basic level, fourth grade students might be asked to find a way to organize a kitchen schedule to serve a large family holiday meal based on such factors as cooking times, oven availability, clean up times, equipment use, etc. (English 2007). They are engaging in modeling when they construct their schedule based on non-overlapping time periods for equipment, paying attention to time constraints. On the other hand, when high school students participate in a discussion to determine the “efficiency” of the packaging for a 12-pack of juice cans, and then use formulas for areas and volumes, calculators, dynamic geometry software, and other tools to create their own packaging and measure efficiency to find the most efficient, they are also engaging in modeling.

Example: Mathematical Modeling.

“Giant’s Shoes.” In a sports center on the Philippines, Florentino Anonuevo Jr. polishes a pair of shoes. They are, according to the Guinness Book of World Records, the world’s biggest, with a width of 2.37 m and a length of 5.29 m. Approximately how tall would a giant be for these shoes to fit? Explain your solution. (Picture: Blum & Ferri 2009, 45.)

Mathematical modeling plays a part in many different professions, including engineering, science, economics and computer science. Professional mathematical modeling involves looking at an often novel real-world problem or situation, asking questions about the situation, creating mathematical representations (“models”) that describe the situation (e.g. equations, functions, graphs of data, geometric models, etc.), computing with or extending these representations to learn something new about the situation, and then reflecting on the information found. Students, even those in the earlier grades, can be encouraged to do the same: when presented with a real-world situation, students can ask questions that lead to applications of mathematics to new and interesting situations and lead to new mathematical ideas (“How could we measure that?” “How will that change?” “Which is more cost effective and why?”).

Mathematical modeling can be seen as a multi-step process: posing the real-world question, developing a model, solving the problem, checking the reasonableness of the solution, and reporting results or revising the model. These steps all work together, informing one another, until a satisfactory solution is found. Thus, the parameters in a linear model such as $f(x) = 0.8x + 1.5$ may need to be altered to better predict the growth of the supply of a product over time based on initial calculations. Or, a simplification that was made earlier in the model formation may need to be revisited to develop a more accurate model.

Blum and Ferri (2009) offer a schematic that describes a typical modeling process:

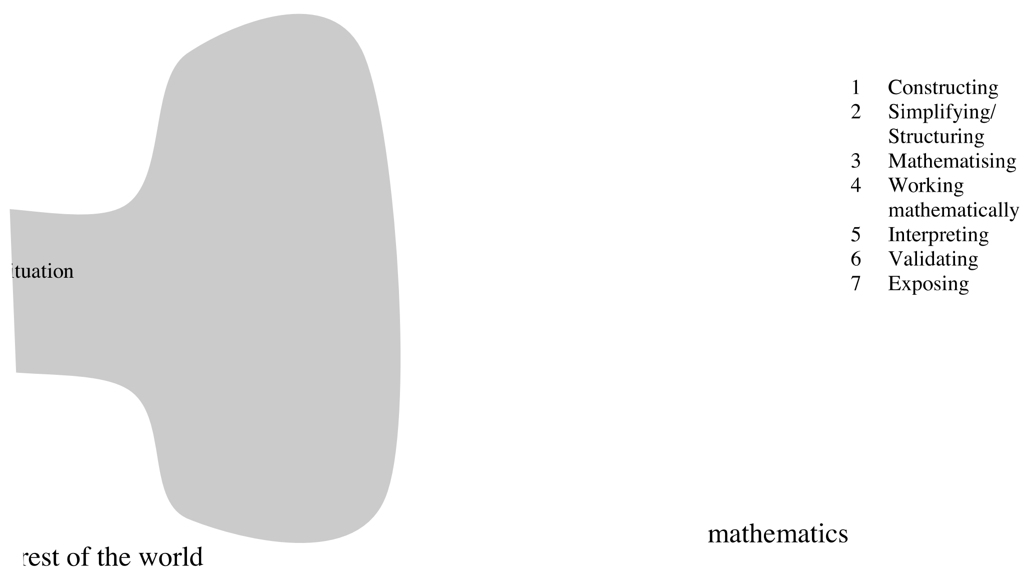

Figure 1: Modeling Cycle (Blum and Ferri, 2009, 46)

In this cycle, the first step is examining the real world and constructing a problem, typically by asking a question. Second, the important objects or aspects of the problem are identified, and simplifications are made if necessary (e.g. ignoring that a juice can is not exactly a cylinder). Next, the situation is “mathematized”: quantities are identified through measurement, relationships among quantities are described mathematically, or data is collected. This is the step of creating a “mathematical model.” Next, the modeler works with their model—solving an equation, graphing data, etc.—and they then interpret and validate their results in the context of the problem. At this step the modeler may need to return to their models and refine them, creating a looping process. Finally, the results of modeling the problem are disseminated.

The role of modeling in teaching the CA CCSSM

Modeling supports the CA CCSSM goals of preparing all students to be college and career ready by teaching them that mathematics is a part of their world and can describe the world in surprising ways. Modeling supports the learning of useful skills and procedures, helps develop logical thinking, problem solving, and mathematical habits of mind, and promotes student discourse and reflective discussion. Lastly, modeling allows students to experience the beauty, structure and usefulness of mathematics.

In contrast with the typical “problem solving” encountered in schools, modeling problems have important mathematical ideas and relationships embedded within the problem context and students elicit these as they work through the problem (English2007, 141). Thus, as opposed to a word problem in which students are simply required to apply a mathematical skill they have just learned in a new context, often in a modeling situation the exact solution path is unclear, and may involve making assumptions that lead students to use a mathematical skill and reflect on whether they were justified in doing so. In addition, modeling problems "necessitate the use of important, yet underrepresented, mathematical processes such as constructing, describing, explaining, predicting, and representing, together with organizing, coordinating, quantifying, and transforming data." (English 2007, 141-142) These are some of the same mathematical processes encapsulated in many of the Standards for Mathematical Practice. Modeling problems "are also multifaceted and multidisciplinary: students' final products encompass a variety of representational formats, including written text, graphs, tables, diagrams, spreadsheets, and oral reports; the problems also cut across several disciplines including science, history, environmental studies, and literature. (English 2007, 141-142)

Current mathematics education literature points to two main uses of modeling in teaching: "modeling as vehicle" and "modeling as content." (See Galbraith 2012.)

- **Modeling as Vehicle:** In this perspective, modeling is seen as a way to provide an alternative setting in which students can learn mathematics. This is the perspective that views modeling as being a motivating way to introduce students to new mathematics or to practice and refine their understanding of mathematics they have already learned. When modeling is seen as a vehicle for teaching mathematics, emphasis is not placed on students becoming proficient modelers themselves.
- **Modeling as Content:** In this perspective, modeling is experienced as *content* in itself. Here, specific attention is placed on the development of students' skills as modelers as well as mathematical goals. With modeling as content, mathematical concepts or procedures as not the sole outcome of the modeling

> activity. "When included as content, modeling sets out to enable students to use their mathematical knowledge to solve real problems, and to continue to develop this ability over time." (Galbraith 2012, 13.)

Both viewpoints of modeling can be included in school mathematics curricula to achieve the complementary goals of student learning of mathematics content and student learning to be modelers. However, the modeling as content approach has the additional goal of specifically helping students develop their ability to address problems in their world, an important aspect of college and career readiness.

As noted in Burkhardt (2006), people model with mathematics from a very early age. "Children estimate the amount of food in their dish, comparing it with their siblings' portions. They measure their growth by marking their height on a wall. They count to make sure they have a 'fair' number of sweets." (Burkhardt 2006, 181.) Zalman Usiskin notes that there is a stark example of mathematical modeling in many classrooms: the grading system. "A student obtains a score on a test, typically a single number. This score is on some scale, and that scale is a mathematical model that ostensibly describes how much the student knows… the problem to model is that we want to know how much the student knows." (Usiskin 2011, 2.) He cites another common example of modeling that doesn't appear to be so: determining how big something is. "Consider an airplane. We might describe its size by its length, its wingspan, its height off the ground, its weight, the maximum weight it can handle and the maximum number of passengers it can handle… We recognize that [one] cannot describe an airplane's size by a single number." (Usiskin 3.) The class may be voting a new class president. Some voting systems allow you to rank your top three candidates and assign different values based on placement (e.g., First = 5 points, Second = 3 points, Third = 1 point). Is this a fair way to determine a winner? These and many other examples show that mathematical modeling occurs from very early on and that modeling questions can arise in many different situations. Thus, there is a unique opportunity in mathematics education to build on this seemingly innate tendency to use modeling to understand the world.

The way to bring this to the classroom is another difficult task altogether. Something to the advantage of the teacher is the CA CCSSM focus on *depth* as opposed to coverage; fewer separate concepts in each grade or course can allow for more time for modeling experiences to allow students to learn those concepts at a deep level. However, several challenges to teaching mathematical modeling will arise, not the least of which include understanding the role of the teacher as well as the role of the students, the availability of modeling curriculum, and support for teachers. Each of these will be discussed in more detail below, but it is clear that modeling with mathematics will be new to many teachers and students and so introducing modeling in a classroom should be done with care and patience.

Example: Modeling in the 4-6 Classroom.

Example: "Holiday Dinner." The three Thompson children, Dan, Sophie and Eva, want to organize and cook a special holiday dinner for their parents, who will be working at the family store from 7AM to 7PM. They will decorate the house, prepare, cook and serve holiday dinner, and they know that they need to carefully plan out a schedule to get everything done on time. The last time they tried something like this, for their parents' wedding anniversary dinner, they created an activity list and a schedule for preparing and cooking the meal. Unfortunately, the schedule they made previously did not work very well. They found that they stumbled around the kitchen wanting to use the same equipment at the same time. The Thompson children realized that they didn't think of all the things they needed to include in their schedule.

The children have decided the menu for their holiday dinner:

- First: Before dinner appetizers (cheese, dip, carrot sticks and crackers)

- Second: Baked turkey, roasted vegetables, and steamed vegetables
- Third: Pavlova[1], ice-cream and fresh strawberries

Dan, Sophie and Eva know their parents will be home at 7PM, and they are all available to begin preparing the dinner at 2:30 PM. They have four and one-half hours to get everything ready! All they need to do is organize a schedule that works better than the wedding anniversary schedule.

Here are some of the things they need to consider:

- How long the turkey will take to cook
- What other items can cook in the oven with the turkey
- When to decorate and set the table
- When to make the pavlova and how long it will take
- How often they need to clean in between the cooking
- How much bench space they have for food preparation
- What food needs to be ready first
- Who will use the equipment and when
- Who will be responsible for what jobs!

In the kitchen, there are two benches to work on, a double sink, a microwave oven, and a stove with four top burners and an oven. The oven is large enough to fit the turkey and one other item at the same time.

Dan, Sophie and Eva need your help! They have so many tasks to complete to be ready to surprise their parents, and they need a reliable schedule. Can you help them do two things?

1. Make a preparation and cooking schedule. Chart what each person will do and when, including the use of kitchen equipment.
2. Write an explanation of how you developed the schedule. They plan to have other surprise celebrations for their parents and want to use your explanation as a guide for making future schedules.

(Adapted from English 2007.)

The role of the teacher

The image of students in a classroom feverishly working on a real-world problem that resulted from a question they themselves asked paints a different picture of the role of the teacher. In teaching modeling, the teacher is seen much more as a guide or

[1] A marshmallow-centered, baked meringue dessert.

facilitator, allowing students to follow a solution path they have come up with and making suggestions and asking questions when necessary. Teachers in the modeling classroom are aware of suitable contexts so that their students have an entry point and can ask appropriate questions to attempt to answer. When using modeling to teach certain mathematical concepts, teachers guide the class discussion towards their goal. Teachers in a modeling classroom move away from a role of manager, explainer, and task-setter, and towards a role of counselor, fellow mathematician, and resource. (Burkhardt, 188.)

Teachers new to modeling may have a difficult time allowing their students to grapple with a difficult mathematical situation. Modeling involves problem solving, and, “Problem solving involves being stuck. If a task does not puzzle us at all, then it is not a problem. It is merely an exercise.” (Abrams 2001, 20.) Teachers will need to keep this in mind and remember that learning occurs as the result of struggling with a difficult concept, and so a certain amount of student struggle is necessary and desirable.

Blum and Ferri (2009) posit some general implications for teaching modeling based on empirical findings. They note that:

1) Appropriate modeling tasks must be provided for students, and a balance between maximal student independence and minimal teacher guidance should be found.
2) Teachers should be familiar enough with tasks that they can support students’ individual modeling routes and encourage multiple solutions.
3) Teachers must be aware of means of strategic intervention during modeling activities.

4) Teachers must be aware of ways to support student strategies for solving problems.

To help with implication (4), they suggest a four-step schematic, simplified from the earlier seven-step diagram, for guiding students' strategies:

Four steps to solve a modelling task ("Solution Plan")

to 1

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Figure 2: Blum and Ferri 2009.

The role of students

The transition to modeling in the classroom may prove difficult for students as well. It is no secret that many mathematics lessons involve the teacher explaining and showing while students imitate the teacher. As Burkhardt notes,

> Most school mathematics curricula are fundamentally imitative—students are only asked to tackle tasks that are closely similar to those they have been shown exactly how to do. This is no preparation for practical problem solving or, indeed,

> non-routine problem solving in pure mathematics or any other field; in life and work, you meet new situations so you need to learn how to handle problems that are *not* just like those you have tackled before. (2006182.)

The transition from passive learner to active learner will pose a major challenge to students who are accustomed to simply mimicking their teacher’s actions. However, this transition is empowering for students; they become powerful problem solvers when they combine reasoning and persistence to solve problems where the outcome is meaningful.

Teachers can help facilitate this transition for students by starting with manageable modeling situations and gradually increasing the complexity of tasks. Students will need to learn in modeling situations that the teacher more than ever is a resource and not simply an “answer-provider;” the students are now responsible for doing the hard work. Through entry-level modeling tasks, students can learn to be investigators, managers, and explainers, and become responsible for their own reasoning and its correctness. Eventually, students can pose their own questions and fully carry out the modeling process. Abrams (2012 46) suggests a “Spectrum of Applied Mathematics” that can be followed when providing tasks that will allow students to ramp up to full modeling. The spectrum below is derived from Abrams’s work. This spectrum should be viewed as just that: a spectrum and not a *ladder*, in the sense that teachers can enter the spectrum in various places according to the needs and abilities of their students.

A Spectrum of Mathematical Modeling (Examples are Suitable for Upper Middle/High School)
Level 9: (Highest Level) Students choose the context and the question. They experience the entire modeling process confronting two or more iterations. The question may be practical or may concern something about which the student is curious.
Level 8: The context is provided by the teacher. Students determine a meaningful question related to the context and use the modeling process to determine an answer. *Example:* Presented with a 12-pack of juice cans (or water bottles), what questions could be asked that would lead to a practical solution?
Level 7: The teacher determines the context and poses the question to be answered. Students determine the relevant variables, make assumptions and choose to simplify or ignore some of the variables. Students will need to justify their decision when making presentations. *Example:* Find a better way to package juice cans.
Level 6: Same as level 7 however the teacher guides students through the process of making assumptions and simplifications. Students develop and apply mathematical models and determine the reasonableness of the solutions. *Example:* Find a better (more efficient) way to package juice cans. The discussion will determine that "efficient" means the ratio of the space used to the space available in the package. Students and teachers will assume the cans are perfect cylinders, restrict the package to single can height, all cans are oriented in the same direction, and the package is a prism with congruent polygon bases (no shrink wrap).
Level 5: The teacher provides a simplified version of a real-life question and context. The problem is rich enough to allow for several solution paths and allow for access to various levels of mathematical background. *Example:* Which package uses the highest percentage of space: a rectangular 12-pack, a triangular 10-pack, a trapezoidal 9-pack or a hexagonal 7-pack? (All are prisms with a height equal to one can or bottle). Students make accurate representations of these packages and may determine the use of space using measurement, algebraic manipulation applied to polygons, or geometric sketchpad software.
Level 4: Students are guided through the solution process that starts with a real-life context and question. The series of questions assures students will follow a particular path and use expected mathematics to solve the problem. Reasonableness of the solution is analyzed. *Example:* Which is more efficient: the hexagonal 7-pack or the triangular 10-pack? Determine the percentage of space used in a hexagonal 7-pack (see picture). (1) The diameter of a can is 6.6 cm. Determine the area of a circle with radius 3.3 cm. (2) Multiply by 7 to determine the total area of seven circles. (3) Divide the hexagon into six congruent triangles and find the area of one of the triangles. (4) Connect the centers of three circles to form an equilateral triangle. (5) etc.
Level 3: A context and question are given. This is a real-world context with a mathematical focus. *Example:* Six cans (circles) are placed together to form a triangle shape. The design engineer needs to find the height of the configuration. Determine the distance from the bottom can to the top can.
Level 2: The context or real-world nature is incidental to the problem. Problem may even be contrived.

Example: Three circles are placed tangent to one another. Calculate the area bounded by the three circles.
Level 1: No real-world context. The question is purely mathematical. *Example:* Calculate the area of a circle with diameter equal to 2.5 inches.

The modeling curriculum

As noted earlier, much of the current mathematics curricula available are imitative in nature. Real-world situations are often only employed as exercises for students to practice mathematics they are currently learning, often in the form of word problems. The preceding discussion already pointed out some features of a modeling curriculum, which include open-ended tasks, complex problems, student independence, and multiple means of sharing results. Abrams (2012 40) suggests some differences between true mathematical modeling situations and mathematical exercises with the following table:

Mathematical Modeling	**Mathematical Exercises**
Unfamiliar	Familiar
Memorable	Forgettable
Relevant	Irrelevant
Many possible correct answers	One right answer
Lengthy	Brief
Complex	Simple
Discovering processes	Following instructions
Open-ended	Closed (goal chosen by teacher)
Cyclic—constant refining	Linear
Doesn’t appear on a particular page	Appears too often and then not enough

Modeling often involves project-based work, which can take place over days, weeks or even months. Modeling problems are unfamiliar and original to students; they are memorable due to students taking such an active role in their learning. They can be whimsical and clever with the potential of being extended to the real world. Modeling tasks have an inherent relevance, as students clearly see applications to the real world. Tasks are not predetermined and often have messier endings than traditional problems, as sometimes students must simply decide whether they have enough information to make a decision. Modeling situations offer great opportunities for cross-disciplinary work, and may include drawing from the sciences for problems and requiring writing reports as a summative activity. Finally, real modeling problems do not come with instructions. Students may take a certain solution path only to find that it did not shed much light on the situation, so that they need to start over along a different path.

Teachers should scrutinize newly developed CA CCSSM-aligned instructional materials with their newfound knowledge of the features of modeling. Teachers may find it necessary to supplement their curricula with rich modeling tasks. See the list of Framework Resources for some well-known references for teaching modeling.

Teachers can even make use of problems from traditional curricula in modeling; a traditional word problem can often be changed into a modeling task by asking what would happen if something about the original problem were changed. Teachers can use their own experience to experiment with developing their own modeling tasks for students.

When the goal is the learning or application of particular mathematics content, the challenge for teachers and curriculum developers will be to select appropriate tasks for investigation using the modeling process. The tasks must involve question formulation based upon authentic real-world contexts that are likely to introduce, develop, or apply the desired mathematics content. Since real-world contexts that are rich are often complicated, simplification in the modeling process will be a critical step. Care must be taken to avoid contrived or overly simplified problems. As students develop a deeper understanding of the mathematical modeling process they should be involved more and more in the question formulation and simplification stages. In addition, students should experience the latter steps in the mathematical modeling process. Real-world questions should lead to real-world solutions. The solutions are examined for reasonableness, does the answer make sense, and usefulness, is the solution applicable to the original situation or is it necessary to revisit and reformulate the model.

Supporting teachers and students

As with any change in instruction, many teachers will benefit from their own professional learning with regards to mathematical modeling. Teachers in a professional learning setting should experience the process of modeling themselves. By modeling, teachers can get a feel for looking at the world through a mathematical lens; they begin to ask questions, notice peculiar situations, and recognize the usefulness of mathematics in the world. Such exposure is certainly the first step in teachers learning to employ modeling in their classrooms.

Teachers will also need experience recognizing, creating and modifying good modeling problems. Consider Usiskin's "reverse given-find" problems. Typically, mathematics word problems are of the "given-find" variety, wherein certain information is known (i.e. given) and students are asked to derive some unknown information (i.e. find x). For example, given the sides of a triangle, students are asked to find its perimeter; given the side lengths of a rectangle, students find its area; given a polynomial, they find its roots; etc. Usiskin suggests reversing these questions: A triangle has perimeter 12 units, what are the possible whole number side lengths of this triangle? A rectangle has area 24 square units, how many rectangles with whole number side lengths have this area? A polynomial has the following roots… can you determine the polynomial? (Usiskin2011, 5) Of course, this is only one way to develop simple open-ended situations and they are mathematical in nature. However, teachers can get started in this way and expand to more complex examples and real-world situations.

[SIDEBAR]

The Modeling Process Is Enhanced By:

1. The facilitative skill of the teacher. The teacher must create a positive and safe environment where student ideas and questions are honored and constructive feedback is given by the teacher and by other students. Students do the thinking, problem solving and analyzing.
2. The content knowledge of the teacher. The teacher understands the mathematics relevant to the context well enough to guide students through questioning and reflective listening.
3. Teacher and student access to a variety of representations, and mathematical tools such as manipulatives and technological tools (dynamic geometry software, spreadsheets, internet, graphing calculators, etc.).
4. Teacher and student understanding of the modeling process. Teachers and students who have had prior experience have better understanding of the modeling process and the use of models.
5. Teacher and student understanding of the context. Background information/experience may be needed and gained through Internet searches, print media, video, pictures, samples, field trips, guest speakers, etc.
6. Richness of the problem to invite open-ended investigation. Some problems invite a variety of viable answers and multiple ways to represent and solve. Some contrived problems may appear to be real-world but are not realistic or cognitively demanding.
7. Context of the problem. Selecting real-world problems is important, and real-world problems that tap into student experience, (prior and future), and interest are preferred.

In addition to experiencing modeling, teachers will need to develop a different set of skills than they may possess. For example, while the ability to conduct student discourse—allowing discussions to unfold in a non-directive but supportive way, allowing students the time to discuss their ideas—is critical for teaching many of the Standards for Mathematical Practice, it is of crucial importance for teaching modeling. Teachers also must develop knowledge of tasks, knowledge of the steps of the modeling process so as to identify student difficulties, and knowledge of intervention strategies.

Teachers' belief systems concerning the teaching of mathematics will need to undergo a shift. As mentioned earlier, the goals of modeling are often not strictly mathematical. Modeling can help students better understand the world and help create a more rounded picture of mathematics for them, in addition to teaching mathematical content. In order to give modeling the time and space needed to implement, teachers will need to reflect on their goals for instruction and must be willing to practice patience and understanding that the modeling process represents a rather large instructional shift.

Finally, as noted in the chapter, "Supporting High Quality Common Core Mathematics Instruction", administrators must allow teachers the time and space to implement new CA CCSSM teaching strategies in the classroom. Many of the Standards for Mathematical Practice (MP) are encompassed in the modeling process itself (e.g. MP.1, MP.2, MP.3, MP.5, MP.6), and so teaching with modeling, both as vehicle and as content, supports teaching the CA CCSSM. Administrators must be aware of this and should be supportive of teacher efforts to include modeling in their instruction, especially in middle and high school classrooms. Administrators should be aware that classrooms engaged in modeling tasks are noisy and messy, with students often working independently and excitedly in groups. Teachers with backgrounds in engineering or the sciences will be a good resource for implementing modeling.

Finally, the role of parents should also be highlighted. Many great ideas in improving mathematics education have found resistance due to the misunderstanding of parents. In the case of mathematical modeling, problems are messier and teachers are not simply showing students what to do and then having them practice. If parents are informed of the goals and process of modeling, then they can better understand their child’s response to, "what did you do in math class today?" Parents must be included in the support structure if CA CCSSM is to be successful.

Modeling in Higher Mathematics

Modeling is considered a conceptual category in the CA CCSSM for higher mathematics. By the time students have gained proficiency in the K-8 standards, their understanding of number and operations, equations, functions, graphing, and geometry is quite solid. In the higher mathematics courses, students will develop these ideas further, especially the notion of function, which can play an important role in modeling. While the function concept is developed more fully, and as students’ repertoire of expressions and equations increases, students are able to work with more challenging modeling situations.

[This will be a sidebar comment.]

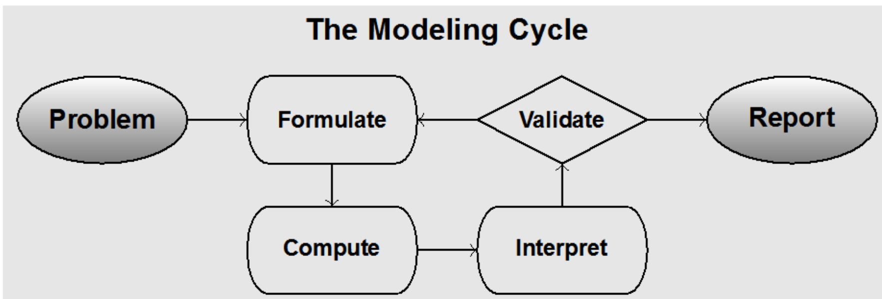
Abrams (2012, 43) describes a modeling problem in which students asked themselves, “How can you eat a peanut butter cup candy in more than one bite and ensure that each bite has the same ratio of chocolate to peanut butter?” After simplifying the peanut butter cup to two cylinders, a peanut butter cylinder embedded in a chocolate cylinder, and simplifying a bite into an arc of a circle intersecting these two cylinders, students went to work trying to discover a formula for the volumes of both chocolate and peanut butter in each bite. The students eventually derive an equation that involves complicated rational expressions contained in square roots, inverse trigonometric functions, and both variables and parameters. They find the ratio in question as a function of the size of the first bite. As Abrams admits, though the problem is not the most important question for humankind to answer, his students were completely engaged with it and quickly discovered how hard the problem actually was.

It is of note that this question can be considered a “whimsical” one. It is a fun question that results in a real-world answer but may not be important in the grand scheme of things. However, many mathematical discoveries and questions were discovered by beginning with a similar whimsical problem, and so such questions should certainly be explored and encouraged.

The first course in higher mathematics is the first place where mathematical modeling is introduced as a conceptual category. “Model with mathematics,” Standard 4 of the Standards for Mathematical Practice, should have been a common experience for students in prior grades. Students should have already had numerous opportunities to apply mathematical models to solve real-world problems. In the first course of higher mathematics, (e.g., Mathematics I or Algebra I) explicit attention will be given to teaching the *process* of mathematical modeling. Students will now learn and practice *all* the steps in the process; they will understand the modeling process is seldom linear and that it often involves revisiting steps in order to formulate a model that is both useful and solvable. The model developed must authentically approximate the real-world context and, at the same time, provide access to students in terms of the mathematics needed to understand the situation or answer the question.

Problems arising from the real world will seldom involve a single content standard. Informed by experience with a broad array of applications, teachers and curriculum developers will find real-world contexts likely to elicit the mathematics in this course.

The CCSSI provides a schematic for modeling at the higher mathematics level:



According to the authors:

> The basic modeling cycle is summarized in the diagram. It involves (1) identifying variables in the situation and selecting those that represent essential features, (2) formulating a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables, (3) analyzing and performing operations on these relationships to draw conclusions, (4) interpreting the results of the mathematics in terms of the original situation, (5) validating the conclusions by comparing them with the situation, and then either improving the model or, if it is acceptable, (6) reporting on the conclusions and the reasoning behind them. (CCSSI, Appendix A of the CCSSM.)

The vision of modeling in the CA CCSSM higher mathematics aligns with much of the discussion in this appendix. The list of resources provided will be useful for those who wish to delve deeper into teaching modeling.

The following table presents some examples of modeling problems suitable for upper middle and higher mathematics courses.

Modeling in Upper Middle and Higher Mathematics Courses.
Example: Linear Functions. There are numerous real-world contexts that can be modeled using linear functions. Situations involving repetitive addition of a constant amount are plentiful. Common contexts such as comparing cost, revenue and profit for a simple business or any context with a fixed and variable component such as a membership fee combined with a monthly maintenance fee or a down payment followed by monthly payments or a beginning amount with constant growth such as simple interest are opportunities to apply linear models. Students may be asked to determine the feasibility of starting a business such as the selling of hot dogs. A teacher may facilitate a class discussion to identify relevant factors, make assumptions, and gather needed information. The class might survey the market to determine a reasonable price such as $2.25 per hot dog. The cost factors, such as ingredients and paper goods, could be simplified and condensed into a single cost-per-hot dog such as $1.10. The total cost will usually include fixed costs such as rents and license. Many towns have street fairs or “Market Night” where a space may be rented for a cost such as $50.00 for a four-hour period. Tables, graphs or equations may be used as models to answer teacher or student questions such as, “how many hot dogs would I need to sell if I need to make a profit of $400?” or “how many hot dogs would I have to sell per hour to break even?” Graphing calculators, computer applications or spreadsheets could also provide powerful models to generalize or extend the investigation.
Example: Exponential Functions. Exponential functions model situations representing a constant multiplier. Situations involving population growth or decay, the elimination of a therapeutic drug in the body, the filtering of harmful pollutants in air or water, compound interest or cell division. The current population of a town is 18,905 individuals. If population is growing at an average rate of 3% each year then when should the population be expected to reach 20,000 individuals? What services, provided by the community, might need to increase and thus be reflected in the town’s budget? The use of an average rate would be a simplification. Perhaps students would rather investigate a high and low rate to project a range of possible population projections.
Example: “Juice Can Packaging.” A problem from the A.R.I.S.E., recorded in The Mathematics Teacher, NCTM, 1997, and adapted here involves the packaging of juice cans. [Note: water bottles can

be substituted for juice cans.] Typically, juice cans are packaged into rectangular prisms in quantities of six, twelve or twenty-four cans. The cans are situated in a rectangular array; space between the cans is wasted. Students are challenged to design a new package that will minimize the amount of space that is wasted.

Students identify variables and assumptions; the most important aspects are used to focus the problem. Students or the teacher may decide to limit the number of cans to a minimum of four and a maximum of twelve with the knowledge that any simplifications may be revisited when the reasonableness of solutions, based on the models, is analyzed. Additional restrictions may include: the packages are prisms with polygon bases, all cans are situated in the same direction, the cans are perfect cylinders, and the cans are not stacked. The last restriction allows students to simplify the models to two dimensions, circles within polygons, since the dimension of height is held constant for all designs.

Students begin the investigation by moving cans around to visualize different arrangements. They represent their designs with careful drawings (circular disks) or using geometric sketchpad technology. Decisions are made about how to determine whether the wasted space is calculated as an absolute area or volume or a percentage of the available space. Students incorporate the Pythagorean Theorem, equilateral and 30-60-90 right triangles, similar triangles, area of polygons, area of circles, tangents to circles, volume of cylinders, ratios and proportions.

Creating a Mathematical Modeling Course for High School

A course in Mathematical Modeling will build upon modeling experiences in previous mathematics courses. A modeling course should allow students to deepen their understanding of the modeling process, apply in new contexts mathematical models already learned, and learn new mathematics content to solve unique real-world problems.

Students with a strong background in mathematical modeling should be able to apply mathematics to understand or solve novel problems in career and college settings. A modeling course should allow students to experience all stages of the modeling process, including problem formation, model-building utilizing a variety of mathematical models, skills and tools for solving the problems, and sufficient analysis to determine if the solution is reasonable or if the model should be revised.

The goal of mathematical modeling is to answer a question, solve a problem, understand a situation, design or improve a product or plan, or make a decision. Mathematical modeling in the school setting includes the additional expectation that students will learn or apply particular mathematical content at a given grade level. If the learning or applying of content standards is the goal, then real-world problems will need to be selected that are likely to imbed the desired mathematics.

Most teachers and students have experienced one path for learning higher mathematics. The traditional pathway is logical, building concept upon concept, increasing in complexity, intent on accumulating tools that may be used to solve problems. Rarely, if ever, are students given the opportunity to solve real-world problems in the way in which they are encountered in life. Typically the “application” problems in textbooks are formulated and presented in the form of exercises with the hope students will buy into the importance of mathematics.

There is another way to learn mathematics that almost no one has experienced in the classroom but most have experienced in everyday life or career. Start with a real-world problem or question, and apply mathematics already learned to a novel situation or learn new mathematics that can be applied to solving the problem.

[SIDEBAR🡻]

Mathematical modeling should be experienced by students and teachers several ways:

- Short, simple real-world problems, questions, or “I wonder…?” scenarios that can be solved using mathematics and with satisfaction, in just a few minutes.
- More complex problems that require additional information acquired through research while also

filtering out extraneous information through the process of simplifying and making assumptions. The search for a problem solution would involve more than a day and more than one iteration through the modeling cycle.

- Extensive problems requiring the development of new models and the learning of new mathematics that might constitute an entire unit of study of relevant math content arising from the single real-world scenario. The unit of exploration can extend the math learned into other contexts so that students begin to see the universality of some mathematical models and processes.
- An entire course developed from extensive problem units. The units may or may not be related to one another but definitely evolve from real-world situations. The less guidance and scaffolding that is offered by the teacher or the curriculum, the more authentic the modeling experience.

Courses in mathematical modeling can be developed to serve a variety of curricular goals. One course may revisit or build upon modeling standards from previous coursework. The emphasis would be on students deepening their understanding and skill by applying previous learning in novel, unique and unfamiliar situations. Another modeling course may be designed to learn new mathematics not addressed in previous courses. It should also be noted that real-world problems are not constrained by content standards and will often incorporate multiple standards with varied depth. A course in mathematical modeling should be an extension of, or supplement to, and not a replacement for, the integration of modeling within all higher-level mathematics courses and pathways.

Financial Literacy is a topic that can find a place in the teaching of the CCSS-M in higher mathematics. Topics for a Mathematical Modeling course could include those dealing with simple cost analysis using linear functions, finding simple and compound interest using exponential functions, finding total cost of payments on a loan, etc. Clearly such topics have relevance to students’ futures and are therefore an important application of modeling to students’ lives.

Example: "Owning a Used Car." How old a car should you buy, and when should you sell it?
The teacher poses this question to a high school class and invites students to research several variables online, including options for financing, total cost of the car, depreciation, gas mileage, and the like. Students organize their information and use mathematics to create an argument for why they would buy a car of the year they have chosen. A modeling situation such as this one could involve proportions, percentages, rates, units, linear functions, exponential functions, and more." (Adapted from Burkhardt 2006, 184.)

A course in Mathematical Modeling should:

- Be in addition to, not a replacement for, the incorporation of mathematical modeling into the fabric of all higher mathematics courses. The conceptual category of Modeling was not intended as a separate course that students may or may not encounter in high school.
- Deepen a student's understanding of, and experience with, all stages of the mathematical modeling process. The course should be about modeling as well as mathematics, and the relationship between the two.
- Allow for sufficient opportunities for students to apply mathematical content already learned to unique problems and contexts.
- Challenge and motivate students to recognize the **need** to learn, and then apply, new mathematics and related models. Once the models and the mathematics are introduced, students are challenged to find other contexts to apply the same or a similar model.
- Progressively allow students more and more freedom and opportunities to formulate their own questions, develop and apply and justify their own mathematical models, and analyze and defend their own conclusions through collaboration and dialog with peers and teachers. (This will be a challenge for teachers to provide the right balance of freedom and support. Students need to struggle for learning to take place but not become so discouraged they quit. The teacher will need to know which scaffolds to use and will need to develop open-ended questions to support and sometimes guide students thinking.)
- Help students and teachers recognize that mathematical modeling is something we all do to some extent every day, by developing two related dispositions: (1) the ability to look at a life situation and wonder how mathematics might be applied to understand or solve the situation and, (2) the ability to look at a mathematical concept and wonder how it might be applied to life experiences.
- Provide opportunities for students to tackle real-world problems of varying complexity, from things as ordinary as figuring out which coupon to use, which phone plan to choose, or how much tip to leave, to things as complex as how to best prepare for a disaster, how to solve a crime, how to rate

products, or how to balance the need for increased energy supplies with the need to protect the environment.

- Allow for the learning of mathematical principles, “big ideas,” concepts, procedures, standard models, and skills in a meaningful setting, to establish meaning and relevance before teaching mathematics whenever possible.

Sample Topic Areas in an Applied Mathematical Modeling Course

The starred (★) standards from the higher mathematics CA CCSSM are listed at the end of this appendix. Each standard could be considered as part of a modeling course and could be combined with other higher mathematics standards when creating a course. The following table offers some sample topic areas that might be explored in an applied modeling course.

Topic Area	Sample Contexts or Problems	Intended Mathematics Content
Linear Functions Part 1 • Making Money • Membership • Choosing Plans	• Simple business models of cost, revenue and profit are excellent contexts for modeling with linear functions. A real-life business incorporates several variables that will have to be simplified. Sample questions include: how many of an item must I sell to break even or reach a target profit? Which item should I sell if I want to make the largest profit? • Which plan should I choose? Phone plans, data plans, health clubs, music clubs, membership benefits, etc.. • Should I buy or lease a car? • Salary, commission, or blend?	• Linear functions expressed in tabular, graphical and symbolic forms. • Systems of linear functions and equations. • Recursive forms with constant addition.
Linear Functions Part 2 • Line of Best Fit	• Begin with a set a data that is approximately linear and formulate questions OR • Begin with a real-world question likely to have a linear relationship and then collect the data through simulation, survey or activity or search for data on the internet. • How many people and/or how much food should we plan for a particular event? • How many tickets or programs should we print for this event based on past trends? • How many schools will we need? How many people might attend? What will be the value of my car in seven years? What do we expect will be the total?	• Linear functions derived from approximately linear data. Use of residuals to determine appropriateness of a model.
Exponential	• Contexts related to growth of populations (people, animal, bacteria, disease) or money (compound interest). Make predictions and/or plans based on anticipated growth of the population or money. • Contexts related to decline/decay of populations such as half-life of OTC or prescription drug or depreciation of money. • Contexts related to filtration such as fans to exhaust particulates from a room or remove pollutant from a water supply. • Perhaps include whimsical problems that have sufficient characteristics that are similar to real-world problems. Experience these problems as emotionally safer variants of the real-world situation.	• Exponential functions expressed in tabular, graphical and symbolic forms. • Equations derived from exponential functions. • Decisions to make about limiting the domain to whole numbers, integers or rational numbers for the base and/or the exponent. • Recursive forms involving a constant multiplier. • Introduce inverse of exponential function informally and identify as a logarithm.
Quadratic	• Contexts related to the Pythagorean Theorem and distance. Possibly explore parabolic presence in satellite dish, telescopes, searchlights, spy listening devices and solar cookers. • Contexts related to the sum of a series. Carpet rolls and paper rolls. • Contexts related to projectile motion. • Contexts related to area. • Contexts related to cost, revenue and profit where price is a linear function.	
Polynomial	• Problems related to volume. Produce a box with maximum volume from a flat piece of cardstock but cutting out the corners.	
Absolute Value	• In a town of parallel and perpendicular streets, what is the best location for a new school, hospital, or mall? • Contexts related to tolerance.	
Probabilistic	• Drug testing and determining the cost to test a pool of samples vs. testing individual blood samples. (If ten blood samples are pooled and tested as one and the results are negative then you have saved the cost of testing nine other samples. If the result is positive then you have to re-test the samples individually or in a smaller group.) • Genetic combinations. Fingerprint and DNA testing	• Expected values for false positives, false negatives,
Mixed	• How is mathematics used to build the code for representing the movement of objects on a screen or within a video game? • Blood spatter patterns in a CSI investigation. • Will an asteroid collide with the earth?	• Quadratic functions (for movement of projectiles affected by gravity such as Angry Birds) • Parametric equations involving time and location in two or three dimensions. • Movements driven or altered by forces and represented by vectors that would then involve

		trigonometry ratios and possible law of sines and cosines.
Polygons	• How do you accurately enlarge or reduce an object? How does scaling affect surface area and weight? • What is the most efficient package design for the packing of cylinders into right prisms with polygonal bases? • Where should sprinklers be placed to optimize water coverage for a lawn or crops? • Any packaging or tiling context using polygon-shaped objects either as the content objects or as the package.	• Similarity, scale factors and dilations • Perimeter and area • Tessellations (rotations, translations, reflections)
Trigonometry • Right Triangles	• How can you estimate the height of a tall object, such as a tower or mountain, when you are prevented from finding the direct distance along the ground? • How do you find indirectly the height of an object using a device to measure angle of inclination or angle of depression? • How do you render accurately in a drawing or within a video game the height of an object tilted at an angle to the viewing plane?	• Right triangle ratios (tangent, sine and cosine)
Circles	• Which size pizza is the best deal? • Contexts involving circular motion including wheels, gears, belt-driven motors	• Area and circumference • Tangents to circles
Volume and Surface Area	• Maximizing the volume of a container while minimizing the surface area (amount of materials needed to make the container). • Pistons and displacement in a combustion engine. • Is King Kong possible? How are surface area, weight and volume affected by enlargement or reduction due to scale factor? • Whimsical: How large is the giant that would fit to the world's largest shoe? • How much liquid would it take to fill the giant Coke bottle on display in Las Vegas?	• Prisms, cylinders, cones, spheres • Scale factors (Length, area, volume ratios)

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Table of CA CCSSM Higher Mathematics Modeling Standards	
N-Q.1	Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.★
N-Q.2	Define appropriate quantities for the purpose of descriptive modeling.★
N-Q.3	Choose a level of accuracy appropriate to limitations on measurement when reporting quantities.★
A-SSE.1	Interpret expressions that represent a quantity in terms of its context.★ *a.* Interpret parts of an expression, such as terms, factors, and coefficients. *b.* .Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret* $P(1+r)^n$ *as the product of P and a factor not depending on P*
A-SSE.3	3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★ a. Factor a quadratic expression to reveal the zeros of the function it defines. b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.★ *c.* Use the properties of exponents to transform expressions for exponential functions. *For example the expression* 1.15^t *can be rewritten as* $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ *to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.*★
A-SSE.4	Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.*★
A-CED.1	Create equations and inequalities in one variable **including ones with absolute value** and use them to solve problems in and out of context, including equations arising from linear functions. **CA**★
A-CED.2	Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.★
A-CED.3	Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.*★
A-CED.4	Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law V = IR to highlight resistance R.*★
A-REI.11	Explain why the *x*-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.★
F.IF.4	For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*★
F.IF.5	Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★
F.IF.6	Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★
F.IF.7	Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★ a. Graph linear and quadratic functions and show intercepts, maxima, and minima. ★

	b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.★ c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.★ d. (+) Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior.★ e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.★
F-BF.1	Write a function that describes a relationship between two quantities.★ a. Determine an explicit expression, a recursive process, or steps for calculation from a context.★ *b.* Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.*★ *c.* (+) Compose functions. *For example, if T(y) is the temperature in the atmosphere as a function of height, and h(t) is the height of a weather balloon as a function of time, then T(h(t)) is the temperature at the location of the weather balloon as a function of time.*★
F-BF.2	Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.★
F-LE.1	Distinguish between situations that can be modeled with linear functions and with exponential functions.★ a. Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals.★ b. Recognize situations in which one quantity changes at a constant rate per unit interval relative to another.★ c. Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another.★
F-LE.2	Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table)★.
F-LE.3	Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function.★
F-LE.4	For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.★
F-LE.5	Interpret the parameters in a linear or exponential function in terms of a context.★
F-TF.5	Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.★
F-TF.7	(+) Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context.★
G-SRT.8	Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.★
G-GPE.7	Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, *e.g., using the distance formula.*★
G-GMD.3	Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.★
G-MG.1	Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder.★
G-MG.2	Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot).★
G-MG.3	Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).★
S-ID.1	Represent data with plots on the real number line (dot plots, histograms, and box plots).★
S-ID.2	Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and

	spread (interquartile range, standard deviation) of two or more different data sets.★
S-ID.3	Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers)★.
S-ID.4	Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve.★
S-ID.5	Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data.★
S-ID.6	Represent data on two quantitative variables on a scatter plot, and describe how the variables are related.★ *a.* Fit a function to the data; use functions fitted to data to solve problems in the context of the data. *Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models.*★ b. Informally assess the fit of a function by plotting and analyzing residuals.★ c. Fit a linear function for a scatter plot that suggests a linear association.★
S-ID.7	Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data.★
S-ID.8	Compute (using technology) and interpret the correlation coefficient of a linear fit.★
S-ID.9	Distinguish between correlation and causation. ★
S-IC.1	Understand statistics as a process for making inferences about population parameters based on a random sample from that population.★
S-IC.2	Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation. *For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of 5 tails in a row cause you to question the model?*★
S-IC.3	Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each.★
S-IC.4	Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling.★
S-IC.5	Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant.★
S-IC.6	Evaluate reports based on data.★
S-CP.1	Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or," "and," "not").★
S-CP.2	Understand that two events A and B are independent if the probability of A and B occurring together is the product of their probabilities, and use this characterization to determine if they are independent.★
S-CP.3	Understand the conditional probability of A given B as P(A and B)/P(B), and interpret independence of A and B as saying that the conditional probability of A given B is the same as the probability of A, and the conditional probability of B given A is the same as the probability of B.★
S-CP.4	Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities. *For example, collect data from a random sample of students in your school on their favorite subject among math, science, and English. Estimate the probability that a randomly selected student from your school will favor science given that the student is in tenth grade. Do the same for other subjects and compare the results.*★
S-CP.5	Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations. *For example, compare the chance of having lung cancer if you are a smoker with the chance of being a smoker if you have lung cancer.*★
S-CP.6	Find the conditional probability of A given B as the fraction of B's outcomes that also belong to A, and

	interpret the answer in terms of the model.★
S-CP.7	Apply the Addition Rule, P(A or B) = P(A) + P(B) – P(A and B), and interpret the answer in terms of the model.★
S-CP.8	(+) Apply the general Multiplication Rule in a uniform probability model, P(A and B) = P(A)P(B|A) = P(B)P(A|B), and interpret the answer in terms of the model.★
S-CP.9	(+) Use permutations and combinations to compute probabilities of compound events and solve problems.★
S-MD.1	(+) Define a random variable for a quantity of interest by assigning a numerical value to each event in a sample space; graph the corresponding probability distribution using the same graphical displays as for data distributions.★
S-MD.2	(+) Calculate the expected value of a random variable; interpret it as the mean of the probability distribution.★
S-MD.3	(+) Develop a probability distribution for a random variable defined for a sample space in which theoretical probabilities can be calculated; find the expected value. *For example, find the theoretical probability distribution for the number of correct answers obtained by guessing on all five questions of a multiple-choice test where each question has four choices, and find the expected grade under various grading schemes.*★
S-MD.4	(+) Develop a probability distribution for a random variable defined for a sample space in which probabilities are assigned empirically; find the expected value. *For example, find a current data distribution on the number of TV sets per household in the United States, and calculate the expected number of sets per household. How many TV sets would you expect to find in 100 randomly selected households?*★
S-MD.5	(+) Weigh the possible outcomes of a decision by assigning probabilities to payoff values and finding expected values.★ a. Find the expected payoff for a game of chance. *For example, find the expected winnings from a state lottery ticket or a game at a fast food restaurant.*★ b. Evaluate and compare strategies on the basis of expected values. *For example, compare a high deductible versus a low-deductible automobile insurance policy using various, but reasonable, chances of having a minor or a major accident.*★
S-MD.6	(+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator).★
S-MD.7	(+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game).★

Resources

- The Shell Centre: http://map.mathshell.org.uk/materials/tasks.php
- COMAP: http://www.mathmodels.org/problems/, http://www.comap.com/Philly/CCSSModelingHB.pdf
- The LEMA Project: http://www.lema-project.org/web.lemaproject/web/dvd_2009/english/teacher.html
- The Dana Center: http://www.utdanacenter.org/mathtoolkit/instruction/activities/models.php